NPS55-82-011

NAVAL POSTGRADUATE SCHOOL

Monterey, California

SIMPLE DEPENDENT PAIRS OF EXPONENTIAL AND

UNIFORM RANDOM VARIABLES

by

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March 1982

Prepared for:
Naval Postgraduate School
Monterey, CA 93940

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55-82-011	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SIMPLE DEPENDENT PAIRS OF EXPONENTIAL AND UNIFORM RANDOM VARIABLES		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. J. Lawrance P. A. W. Lewis		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		12. REPORT DATE March 1982
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

bivariate exponential
bivariate uniform
random coefficient
linear function
antithetics
negative dependency
correlation
Spearman
simulation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

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Simple Dependent Pairs of Exponential and Uniform Random Variables

by

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Summary

A random-coefficient linear function of two independent exponential variables yielding a third exponential variable is used in the construction of simple, dependent pairs of exponential variables. By employing antithetic exponential variables, the constructions are developed to encompass negative dependency. By employing negative exponentiation, the constructions yield simple multiplicative-based models for dependent uniform pairs. The ranges of dependency allowable in the models are assessed by correlation calculations, both of the product moment and Spearman types; broad ranges within the theoretically allowable ranges are found. Because of their simplicity, all models are particularly suitable for simulation and are free of point and line concentrations of values.

Key Words: BIVARIATE EXPONENTIAL, BIVARIATE UNIFORM, RANDOM COEFFICIENT, LINEAR FUNCTION, ANTITHETICS, NEGATIVE DEPENDENCY, CORRELATION, SPEARMAN, SIMULATION

6/1/82

Contents

1. BACKGROUND

This paper introduces some models for pairs of dependent exponential random variables constructed as random-coefficient linear functions of pairs of independent exponential random variables. The transformations used occurred first in connection with time series models in exponential variables (see Lawrance and Lewis [1981] for details of this work). These models indicate that a specific random-coefficient linear function of independent exponential variables leads to further univariate and multivariate exponential variables, somewhat analogously to linear functions of independent Gaussian variables leading to further univariate and multivariate Gaussian variables. Negative exponentiation of the transformation then leads directly to a random-power multiplicative transformation, and it is found that this can similarly be employed in constructing pairs of dependent uniform random variables from pairs of independent uniform variables. Both transformations are particularly suited to simulation work, the pairwise uniform giving a base from which to transform to many other bivariate distributions.

This paper then presents results on both dependent exponential pairs and dependent uniform pairs. The transformations used are necessarily limited to situations of positive dependency, but broadening to negative dependency is achieved by employing simple antithetic operations in the constructions. The distributions given are free from point and linear concentrations of probability. If these are required, as is sometimes the case in reliability studies, the concentrations of probability can be put in by using mixtures.

There are, of course, many earlier bivariate exponential and uniform distributions; for instance, the work of Marshall and Olkin [1967] which involves choosing the minimum of two exponentials, and which can easily be

simulated, but which always has a line-concentration of probability. Downton [1970] and Hawkes [1972] developed bivariate exponentials employing the idea of bivariate geometric compounding of independent exponentials; in simulation this would require simulation of bivariate geometrics. Gaver [1972] proposed a bivariate exponential specifically for negative dependency, also allowed in Hawkes' extension, and which in simulation would need both the operations of choosing minima and ordinary geometric compounding; further, it is intrinsically non-reversible, unlike most bivariate exponential models.

Other types of bivariate exponentials could clearly be obtained via the inverse probability-integral transform of other standard bivariate distributions; bivariate uniforms in this category are specifically considered by Barnett [1980]. All these methods involve a fair amount of complication, particularly the inverse probability integral transform, and limited flexibility. The methods proposed in this paper seem simpler than all previous methods at least in the major respect that each bivariate dependent pair is obtained by simple random-coefficient linear transformation of a corresponding independent pair; furthermore they are flexible in their dependency properties, and easy to simulate.

In summary, unless very specific modelling or structural detail is required, e.g. the linear regression of Downton's model, the present models offer the following advantages:

(i) The constructions are very simple.

(ii) They require only two independent exponential random variables and one or two binary random variables for their construction.

(iii) There is a broad range of the attainable dependency, as measured linearly by the product-moment correlation or monotonically by the Spearman correlation.

(iv) The models are analytically tractable so that, in some cases, closed form joint probability density functions and regressions can be obtained.

The scheme of the following sections of the paper is to examine the properties of the very simplest, one parameter, models (EP1 and UP1) in detail. The properties include product-moment and Spearman-type correlations, joint probability functions and regressions. These models have a broad but, positive correlation range which is then extended to negative correlation by using antithetic pairs. In later sections models with more parameters which can attain correlations right up to the theoretical bounds are introduced, but developed in less detail. The most general of these, the EP+ and EP- models, allow the complete range of possible correlations for bivariate exponential pairs and the possibility of accommodating other structural details; they include all the other exponential models. Parallel development is given for bivariate uniform pairs.

2. ONE PARAMETER POSITIVELY DEPENDENT EXPONENTIAL MODEL: EP1 MODEL

The random coefficient linear function referred to in Section 1 is central to this work and is presented first. Let a pair of independent, identically and exponentially distributed random variables be denoted by (E_1, E_2); for simplicity, unit means are assumed throughout. Let I be a binary random variable, independent of (E_1, E_2), with distribution given by $P(I = 0) = \beta$, $P(I = 1) = 1 - \beta$. Then the random variable X , given by

$$X = \beta E_1 + I E_2 , \tag{2.1}$$

has an exponential distribution with unit mean; this can be seen by taking moment generating transforms,

$$\phi_X(t) = E\{\exp(-tX)\} = \frac{1}{1+\beta t} \{\beta + (1-\beta) \frac{1}{1+t}\} = \frac{1}{1+t} \tag{2.2}$$

The result can be extended by taking E_2 to be a random-coefficient linear function of a further pair of exponential random variables independent of E_1 . However, (2.1) is all that is required here to construct bivariate pairs of exponential random variables; the result (2.1) is implicit in the construction of exponential time series models evolving out of the EAR(1) model of Gaver and Lewis [1980].

The EP1 Model

The first and simplest bivariate exponential distribution of the paper is obtained by interchanging E_1 and E_2 in (2.1) to give the second member of a pair, and hence produce

$$X_1 = \beta E_1 + \mathbb{I}E_2 ,$$

(EP1) (2.3)

$$X_2 = \beta E_2 + \mathbb{I}E_1 .$$

These will be called the EP1(β) pair; the pair is clearly dependent because of the common E_1, E_2 and I . There are clear generalizations obtained by using different β's and I's for X_1 and X_2 which will be considered later.

The dependence of (X_1, X_2) can be assessed by their (product moment) correlation coefficient as

$$\begin{aligned} \mathrm{Corr}(X_1, X_2) &= E\{(\beta E_1 + IE_2)(\beta E_2 + IE_1)\} - E(\beta E_1 + IE_2)E(\beta E_2 + IE_1) \\ &= 3\beta(1-\beta) , \end{aligned} \tag{2.4}$$

which is sketched in Figure 1. The correlation thus has the non-negative range (0, 0.75) with maximum value 0.75 at $\beta = 0.5$ and symmetry about $\beta = 0.5$. The product moment correlation emphasizes linear dependency, taking maximal values of $\pm$ 1 if and only if the variables are linearly related which is not the case here; a more general and more appropriate measure of non-linear dependency, Spearman's correlation, will be considered in Section 3.

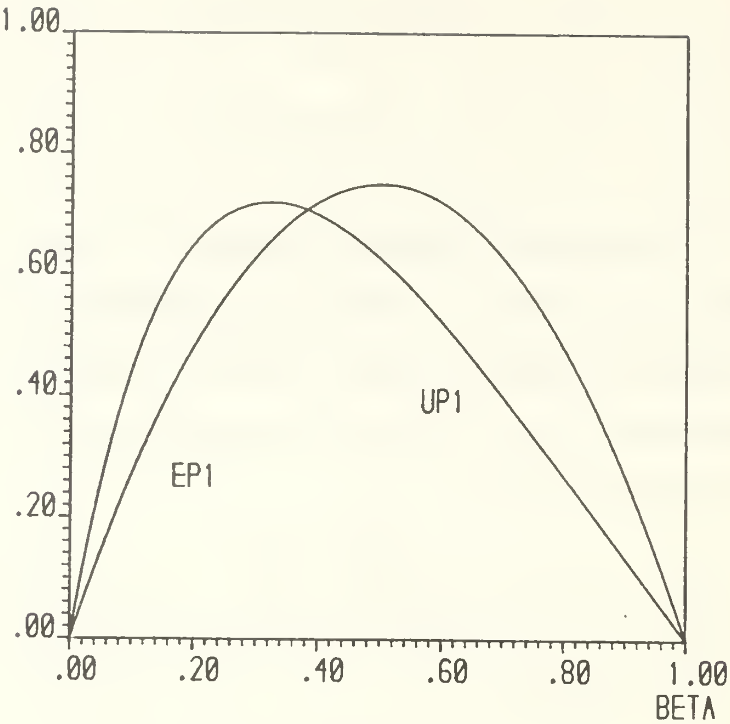

Figure 1. Correlation functions for the EP1(β) model and the UP1(β) model as functions of β; both models exhibit only positive dependence.

The joint probability density of X_1 and X_2 can be obtained by considering the probability element for X_1 and X_2 in terms of $\beta E_1 + IE_2$ and $\beta E_2 + IE_1$. On letting I take its values 1 and 0 with probabilities $1 - \beta$ and β this becomes

$$\begin{aligned}&(1-\beta)P\{(\beta E_1 + E_2) \in (x_1, x_1 + dx_1), (\beta E_2 + E_1) \in (x_2, x_2 + dx_2)\} \\ &\qquad + \beta P\{\beta E_1 \in (x_1, x_1 + dx_1\}P\{\beta E_2 \in (x_2, x_2 + dx_2)\} .\end{aligned} \tag{2.5}$$

The first line of (2.5) can be expressed as a probability element statement in terms of E_1 and E_2 separately, and using the Jacobian, $(1-\beta^2)$, of the transformation from E_1 and E_2 leads to the joint pdf of X_1 and X_2 as

$$f_{X_1,X_2}(x_1,x_2) = (1-\beta)1(x_1 > \beta x_2,\ x_2 > \beta x_1) f_{(1-\beta^2)E_1}(x_2-\beta x_1)$$

$$\times f_{(1-\beta^2)E_2}(x_1-\beta x_2)(1-\beta^2) + \beta f_{\beta E_1}(x_1) f_{\beta E_2}(x_2) \ , \qquad (2.6)$$

where 1( ) is an indicator function taking the value 1 if $x_1 > \beta x_2$ and $x_2 > \beta x_1$ and 0 otherwise, and f denotes the density of the exponential variable in its suffix. Simplifying (2.6) then gives the desired result

$$f_{X_1,X_2}(x_1,x_2) = 1(\beta x_2 < x_1 < \beta^{-1} x_2)(1+\beta)^{-1} \exp\{-(x_1+x_2)/(1+\beta)\}$$

$$+ \beta^{-1} \exp\{-(x_1+x_2)/\beta\} \ , \qquad 0 \le \beta \le 1;\ x_1, x_2 \ge 0 \ . \qquad (2.7)$$

The joint density thus has a wedge shaped sector of relatively high density ($\beta x_2 < x_1 < \beta^{-1} x_2$) superimposed over an independent bivariate exponential density. This function is isometrically plotted in Figure 2 for $\beta = 0.5$. For low values of β the raised section spreads out over the quadrant and

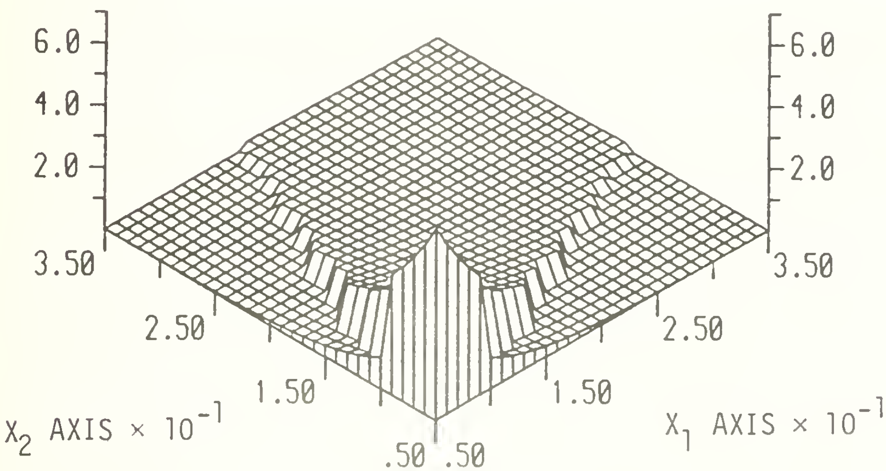

Figure 2. Isometric plot of the joint probability density function of the positively dependent EP(β) model for $\beta = 0.5$. This value of β gives the maximum attainable correlation of 0.75; the figure has been truncated for both variables at 0.5.

leads to independence at $\beta = 0$; for high values of β the raised section diminishes towards nothing and again gives independence at $\beta = 1$.

This joint density function could be used to obtain a maximum likelihood estimate of β from a bivariate sample; note that a unique moment estimator can not be obtained from the correlation (2.2).

The joint moment generating function of the (X_1,X_2) pair, $\phi_{X_1,X_2}(t_1,t_2)$, is easily obtained via the EP1 defining equation (2.3); thus

$$\begin{aligned}\phi_{X_1,X_2}(t_1,t_2) &= E\{\exp(-t_1X_1 - t_2X_2)\}\\ &= E\{\exp[-(\beta t_1 + It_2)E_1 - (\beta t_2 + It_1)E_2]\}\\ &= \beta(1 + \beta t_1)^{-1}(1 + \beta t_2)^{-1}\\ &\quad + (1-\beta)(1 + \beta t_1 + t_2)^{-1}(1 + t_1 + \beta t_2)^{-1} . \end{aligned} \tag{2.8}$$

Differentiating with respect to t_2 at $t_2 = 0$ and inverting with respect to t_1 allows the regression of X_2 on X_1 to be obtained as

$$E(X_2|X_1 = x) = \beta x + \beta + 1 - (1 + x/\beta)\exp\{-(\beta^{-1} - 1)x\} . \tag{2.9}$$

This conditional expectation will be fairly linear except for small x , and the asymptotic conditional variance will be linear for large x . Note from the joint moment generating function (2.8) that the pair (X_1,X_2) is reversible, $\phi_{X_1,X_2}(t_1,t_2) = \phi_{X_1,X_2}(t_2,t_1)$, in particular, $E(X_1|X_2 = x) = E(X_2|X_1 = x)$. An aspect of this reversibility is that $P\{X_2 > X_1\} = 0.5$ which can be shown directly from the definition (2.3).

3. THE NEGATIVE EXPONENTIAL TRANSFORMATION TO UNIFORMITY: UP1 MODEL

It is well known that for an exponential random variable X of unit mean, both e^{-X} and $1-e^{-X}$ are uniform random variables, distributed over (0,1). These results are of double interest here. Firstly, application of the transform to both components of a dependent exponential pair will lead to a pair of dependent uniforms, and secondly, which may be more important to stress, the product-moment correlation in this pair of uniforms is a population analogue of Spearman's rank correlation for the original dependent exponential pair. This was demonstrated generally by Kruskal [1958]; he showed that the product-moment correlation in the transformed pair of dependent uniforms, could be linearly related to the concordance (quadrant) probability

$$P\{X_1 - X_2)(Y_1 - Y_3) > 0\} ,$$

where $(X_i, Y_i\colon\ i = 1,2,3)$ are three independent pairs with the untransformed bivariate distribution. Estimation of this probability was based on constructing the empirical joint distribution of an observed sample, and led Kruskal to estimating the required linear function of the concordance probability by Spearman's correlation. Thus, we have the useful idea that product-moment correlation in a bivariate distribution after its transformation to uniform marginals represents a population analogue of Spearman's correlation for the untransformed bivariate distribution. As such, it is a suitable measure of monotonic dependency for a bivariate distribution. In particular, the Spearman correlation is a suitable measure for dependency in bivariate non-Gaussian distributions because of their intrinsic non-linearity, and the consequent limitations of the product moment correlation.

General Results for the Transformation

In the case of an exponential pair (X_1,X_2) of unit means, the preferred transformation to a uniform pair is given by

$$Y_1 = 1 - \exp(-X_1), \quad Y_2 = 1 - \exp(-X_2) . \tag{3.1}$$

The correlation and regression for this pair are easily obtained from the joint moment generating function, $\phi_{X_1,X_2}(t_1,t_2) = E\{\exp(-t_1X_1 - t_2X_2)\}$ of (X_1,X_2). First for the correlation we have

$$E\{Y_1Y_2\} = E\{(1-\exp(-X_1)(1-\exp(-X_2)\} = \phi_{X_1,X_2}(1,1) , \tag{3.2}$$

and thus the correlation result

$$\mathrm{Corr}(Y_1,Y_2) = 12\phi_{X_1,X_2}(1,1) - 3 . \tag{3.3}$$

For the regression of Y_1 on Y_2, first write it in terms of X_1 and X_2 as

$$E(Y_1|Y_2 = y) = E\{1-\exp(-X_1)|X_2 = -\log(1-y)\} . \tag{3.4}$$

Now setting $x = -\log(1-y)$ and denoting the conditional p.d.f. of $X_1|X_2$ by $f_{X_1|X_2}(x_1|x)$, the regression is

$$E[Y_1|Y_2 = 1-\exp(-x)] = \int_{x_1=0}^{\infty} [1-\exp(-x_1)]f_{X_1|X_2}(x_1|x)dx_1 . \tag{3.5}$$

The required result is obtained after multiplying both sides by $\exp(-tx)$ and integrating x over $(0,\infty)$, so leading to

$$\int_0^{\infty} \exp(-tx)E[Y_1|Y_2 = 1 - \exp(-x)]dx = t^{-1} - \phi_{X_1,X_2}(1,t-1) . \tag{3.6}$$

Thus, the regression of the uniforms follows from inverting the joint moment generating $\phi_{X_1,X_2}(1,t-1)$ of the exponential pair with respect to t as a function of x, and then replacing x by $-\log(1-y)$.

The UP1 Model

The negative exponential transformation to uniformity is now exemplified using the EP1(β) model. To obtain a pair of independent uniform random variables from a pair of independent exponentials it is simplest here to use the monotonic decreasing version of (3.1), that is

$$U_1 = \exp(-E_1)\ , \qquad U_2 = \exp(-E_2)\ . \tag{3.7}$$

It follows then that the dependent uniform pair derived from the EP1(β) exponential model via (3.1) is given by

$$\text{(UP1)} \qquad Y_1 = 1-U_1^{\beta}\,U_2^{I}\ , \qquad Y_2 = 1-U_1^{I}\,U_2^{\beta}\ , \tag{3.8}$$

where I retains its definition from Section 2. The random power multiplicative aspect of the model is evident. The pair (Y_1,Y_2) of (3.8) will be termed the UP1(β) model; the correlation of this uniform pair follows immediately from the joint moment generating function of (X_1,X_2) as given at (2.8), or by direct calculation, and is

$$\mathrm{Corr}(Y_1,Y_2) = 3\beta(1-\beta)\,\frac{8+7\beta+\beta^2}{(1+\beta)^2(2+\beta)^2}\ . \tag{3.9}$$

This function of β is illustrated in Figure 1; its maximum value is 0.72 for $\beta = 0.32$. The joint density of Y_1 and Y_2 may be obtained by considering

$$P(U_1^\beta U_2^I \le y_1,\ U_2^\beta U_1^I \le y_2)$$

$$= \beta P(U_1^\beta \le y_1) P(U_2^\beta \le y_2) + (1-\beta) P(U_1^\beta U_2 \le y_1,\ U_1 U_2^\beta \le y_2) \ . \tag{3.10}$$

Density for the last probability is restricted to the leaf shaped region $(y_2^{1/\beta} \le y_1 \le y_2^\beta,\ 0 \le y_2 \le 1)$, and the final result takes the form

$$f_{Y_1,Y_2}(y_1 y_2) = 1(y_2^{1/\beta} \le y_1 \le y_2^\beta)(1+\beta)^{-1}(y_1 y_2)^{-\beta/(1+\beta)} + \beta^{-1}(y_1 y_2)^{(1-\beta)/\beta}$$

$$(0 \le y_1, y_2 \le 1;\ 0 \le \beta \le 1) \ . \tag{3.11}$$

The density is isometrically plotted in Figure 3 for $\beta = 0.32$, the value giving maximum product moment correlation.

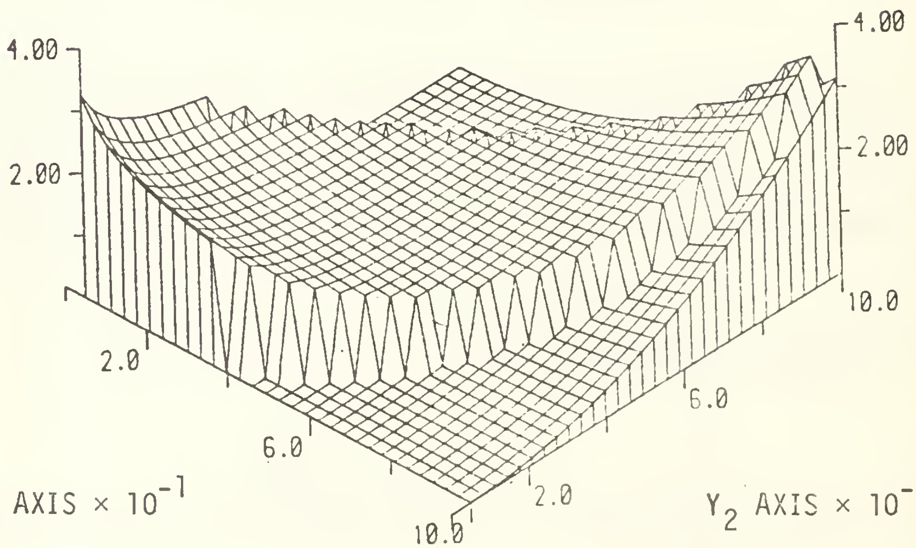

Figure 3. Isometric plot of the joint probability density function of the positively dependent UPI(β) model for β = 0.32 . This value of β gives the maximum attainable correlation of 0.75.

Regression for the UP1 model is most easily obtained via (3.6) using the joint moment generating function of X_1 and X_2 given at (2.8) and takes the form

$$E(Y_2 | Y_1 = y)$$

$$= 1 - (1+\beta)^{-1}(1-y)^{(1-\beta)/\beta} + (2+\beta)^{-1}(1-y)^{(2-\beta)/\beta} - (2+\beta)^{-1}(1-y)^{\beta} . \qquad (3.1$$

The regression has initial value $\beta/(1-\beta)$ which is always less than 0.5 and curves upwards towards 1.0 in a near monotonic concave way. For $\beta = 0$ and $\beta = 1$ it has the value 1/2 which is correct since Y_2 and Y_1 are then independent and $E(Y_2) = 1/2$. Note that the UP1 pair is, like the EP1 pair, reversible, in particular $E(Y_2|Y_1 = y) = E(Y_1|Y_2 = y)$. Also $P\{U_1 > U_2\} = 0.5$; this can be seen directly from the definition (3.8).

4. NEGATIVE DEPENDENCY USING AN ANTITHETIC APPROACH: EP2, UP2 MODELS

The antithetic transformation of any random variable X is found by first expressing it as a function (the inverse probability integral) of a uniform (0,1) variable U, and then by taking the same function of $1-U$. For any nonnegative random variable X, the antithetically transformed variable $a(X)$ has the same marginal distribution as X and is maximally negatively correlated with X; note that the Spearman correlation of X and $a(X)$ is minus one, but their product moment correlation will not necessarily be as negative as this. In the exponential case,

$$a(X) = -\log(1-\exp(-X)) \tag{4.1}$$

and the maximum negative product moment correlation is - 0.6449, as is well known after Moran [1967]. As a pair of exponential random variables, X and $a(X)$ lie on the curve

$$e^{-x} + e^{-a(x)} = 1$$

and so are highly degenerate in the sense that X completely determines $a(X)$.

The EP2 Model

The aim here is to construct negatively dependent exponential pairs which are not, like the antithetic pair, partially or completely degenerate; the random-coefficient linear transformation (2.1) allows us to do this while still making use of antithetic pairs. The general approach is illustrated with respect to the EP1 model; in (2.3) the E_1 and E_2 of X_2 are replaced by their antithetic versions $a(E_1)$, $a(E_2)$, and I in X_1 and X_2 is replaced by I_1 and I_2 respectively, these variables having the same marginal distribution as I. Thus the suggested negatively dependent model has the form

$$X_1 = \beta E_1 + I_1 E_2 ,$$

(EP2) (4.2)

$$X_2 = \beta a(E_2) + I_2 a(E_1) ,$$

and the expression for its correlation is given by

$$\mathrm{Corr}(X_1,X_2) = 2\beta(1-\beta)(-0.6449) + \mathrm{Cov}(I_1,I_2) . \tag{4.3}$$

This suggests that maximum negative correlation of X_1 and X_2 requires maximum negative covariance of I_1 and I_2. For fixed β, this will be achieved for minimum $E(I_1,I_2)$, or equivalently minimum $p = P(I_1 = 1, I_2 = 1)$. Consideration of the general joint distribution of I_1, I_2, that is

$I_1 =$	1	0	
$I_2 = 1$	p	$1-\beta-p$	$1-\beta$
0	$1-\beta-p$	$2\beta-1+p$	β
	$1-\beta$	β	1

, (4.4)

shows that, since the probabilities must be non-negative, the (0,0) term constrains p to the minimum value of $1-2\beta$ if $\beta \leq \frac{1}{2}$ and 0 if $\beta \geq \frac{1}{2}$. The implied joint distribution is, not surprisingly, the antithetic one, and it has covariance $-\beta^2$ for $0 \leq \beta < \frac{1}{2}$ and $-(1-\beta)^2$ for $\frac{1}{2} \leq \beta \leq 1$. Thus, going back to (4.2), the model with $I_1 = I$, $I_2 = a(I)$ is of most interest in connection with negative dependence and will be called the EP2(β) model; its correlation is given by

$$\mathrm{Corr}(X_1,X_2) = \begin{cases} -2\beta(1-\beta)(0.6449) - \beta^2 & 0 \leq \beta \leq \frac{1}{2} , \\ -2\beta(1-\beta)(0.6449) - (1-\beta)^2 & \frac{1}{2} \leq \beta \leq 1 . \end{cases} \tag{4.5}$$

This function is sketched in Figure 4. The minimum value is -0.5724 at $\beta = 0.5$ which, considering the model is never degenerate, is fairly satisfactory. (The Gaver model [Gaver, 1972] is also not degenerate and has a

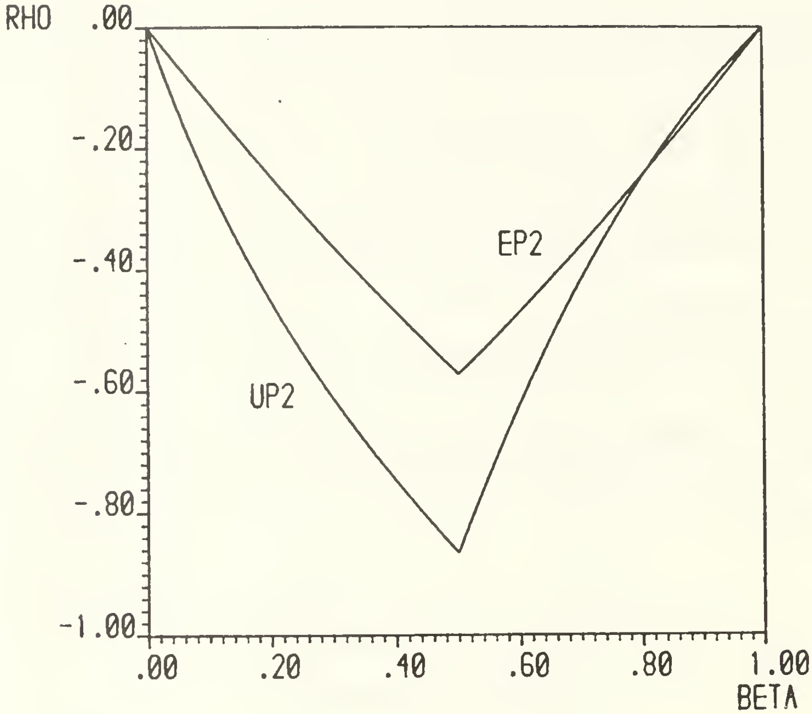

Figure 4. Correlation functions for the EP2(β) model and the UP2(β) model as functions of β; both models exhibit negative dependence.

minimum correlation of -0.5). A simulation of EP2(β) pairs at the $\beta = 0.5$ value is given in Figure 5; the scatter is best described as a non-degenerate emulation of the antithetic concentration along the curve $e^{-x} + e^{-y} = 1$. Weakening of the dependency as the parameter β approaches 0 or 1 fades this characteristic scatter into the blander independent exponential picture. The scatter in Figure 5 should be compared to the isometric plot of the joint density in Figure 3.

The joint moment generating function of X_1 and X_2 requires the joint moment generating function of E_i and $a(E_i)$ which is straightforwardly evaluated as

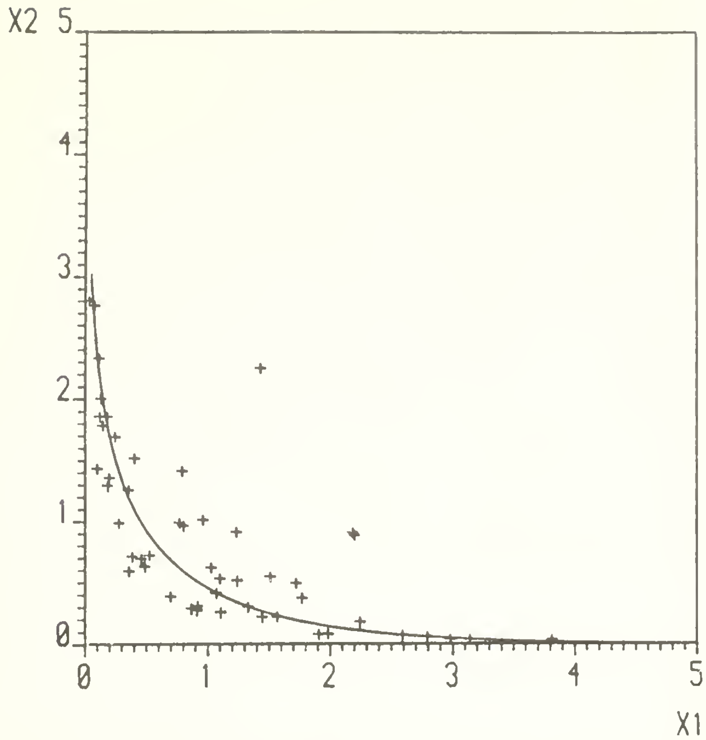

Figure 5. Scatter plot of 50 simulated pairs of observations from the negatively dependent EP2(β) model for β = 0.5: the solid line represents the curve relating the antithetic bivariate exponential pairs.

$$\phi(t_1,t_2) = \int_0^1 x^{t_1}(1-x)^{t_2}dx \ , \tag{4.6}$$

which is the Beta function $B(1 + t_1, 1 + t_2)$. Also required is the joint distribution $p_{ij} = P\{I = i, a(I) = j\}$, $i, j = 1,0$, available as remarked previously following (4.4); hence the result, via (4.2),

$$\begin{aligned}\phi_{X_1,X_2}(t_1,t_2) &= \\ &= p_{11}\phi(\beta t_1,t_2)\phi(t_1,\beta t_2) + p_{10}\phi(\beta t_1,t_2)(1+\beta t_2)^{-1} \\ &\quad + p_{01}\phi(t_1,\beta t_2)(1+\beta t_1)^{-1} + p_{00}(1+\beta t_1)^{-1}(1+\beta t_2)^{-1} \ .\end{aligned} \tag{4.7}$$

Actually the model (4.2) works for any independent bivariate exponential pairs (E_1, E_1') and (E_2, E_2'). However, the antithetic pair is simplest to generate and gives the greatest negative correlation in the (X_1, X_2) pair.

The UP2 Model

Thus far we have the EP2 model giving a broad range of negative dependency for exponential pairs; this is now transformed to the corresponding uniform pair giving the UP2 model. By negative exponentiation of (4.2) and use of (3.1) and (3.6) we have

$$\text{(UP2)} \qquad Y_1 = 1 - U_1^{\beta} U_2^{I}, \quad Y_2 = 1 - (1-U_1)^{a(I)}(1-U_2)^{\beta} . \tag{4.8}$$

The correlation of Y_1 and Y_2 may be evaluated directly from (4.8) or by use of (4.7) and (3.3) and gives the result

$$\operatorname{Corr}(Y_1, Y_2) = \begin{cases} -3\beta(4+5\beta+6\beta^2+\beta^3)/[(1+\beta)^2(2+\beta)^2] & 0 \le \beta \le \tfrac{1}{2} , \\ -3(1-\beta)(2+3\beta-\beta^2)/[(1+\beta)^2(2+\beta)] & \tfrac{1}{2} \le \beta \le 1 . \end{cases} \tag{4.9}$$

A graph of this correlation function is given in Figure 2; its maximum negative value is -13/15 = -0.8667 which occurs at $\beta = 0.5$; this indicates for the EP2 model a good negative range to the Spearman correlation, but no ultimate degeneracy. Bivariate scatters from the UP2 model show a strong degree of linearity compared to the near antithetically curved scatter of the corresponding exponential EP(2) pair in Figure 5. Again, the asymmetry of the dependency about $\beta = 0.5$ in the EP2 model is more evident from the Spearman correlation of the UP2 model. Negative dependency is considerably stronger for $\beta = 0.5 - \delta$ than the corresponding $\beta = 0.5 + \delta$. This parallels the EP1 model in which positive dependency is stronger for $\beta = 0.5 - \delta$ than

for $\beta = 0.5 + \delta$; these remarks are borne out in scatters for various values of β . Finally, it should be noted that the antithetic choice $(I, a(I))$ is best in regard to obtaining negative dependency in the UP2 model, as was found for the EP2 model. Regressions are again not easily tractable.

Positive and Negative Dependence

Extension of the bivariate models given above to models which exhibit both positive and negative product-moment correlation is achieved quite simply by mixture arguments. For example, combining (2.3) and (4.2) let

$$X_1 = \beta E_1 + I_1 E_2$$

$$X_2 = \begin{cases} \beta E_2 + I_1 E_1 & \text{w.p.} \quad p , \\ \\ \beta a(E_2) + I_2 a(E_1) & \text{w.p.} \quad 1-p . \end{cases} \tag{4.10}$$

This uses the fact that a probability mixture of two identically distributed random variables (possibly dependent) has that same identical distribution. These two parameter model contains both the EP1 model ($p = 1$) and the EP2 model ($p = 0$) and has product moment correlation ranging between -0.5724 and 0.75.

5. TWO PARAMETER VERSIONS OF THE MODELS: EP3, UP3, EP4, UP4 MODELS

Dependency in the models has thus far been determined by one parameter β , but it may be desirable to have an additional parameter in the model so that two aspects of the joint distribution can be modelled; for instance, one dependency measure, such as the product-moment correlation, and a probability statement, such as $P(X_1 > X_2)$, which in the EP1 model was equal to 0.5 for all β . Two approaches will be discussed rather briefly. One allows β in the randomly linear combination of E_1 and E_2 to be replaced by β_1 for X_1 and β_2 for X_2 , while in Section 6 a two parameter randomly-linear operation is used: this stems from the NEAR(1) model of Lawrance [1980], Lawrance and Lewis [1981].

The EP3 Model

The first two parameter model, EP3, is defined by the equations

$$\text{(EP3)} \qquad \begin{aligned} X_1 &= \beta_1 E_1 + I_1 E_2 , \qquad & 0 \leq \beta_1 \leq 1 , \\ X_2 &= \beta_2 E_2 + I_2 E_1 , \qquad & 0 \leq \beta_2 \leq 1 . \end{aligned} \tag{5.1}$$

The marginal distributions of I_1 and I_2 must correspond to the different β parameters so that $P\{I_1 = 0\} = 1 - P\{I_1 = 1\} = \beta_1$ and $P\{I_2 = 0\} = 1 - P\{I_2 = 1\} = \beta_2$. The product moment correlation of the model is given by

$$\mathrm{Corr}(X_1,X_2) = \beta_1(1-\beta_2) + \beta_2(1-\beta_1) + \mathrm{Cov}(I_1,I_2) \cdot \tag{5.2}$$

The general joint distribution for (I_1,I_2) is the obvious generalization of (4.4); the value of p_{11} for maximum positive dependence is now $\min(1-\beta_1,1-\beta_2)$. Hence the result

$$\mathrm{Corr}(X_1,X_2) = \begin{cases} 2\beta_1(1-\beta_2) + \beta_2(1-\beta_1) & 0 \le \beta_1 \le \beta_2 \le 1 , \\ \beta_1(1-\beta_2) + 2\beta_2(1-\beta_1) & 0 \le \beta_2 \le \beta_1 \le 1 . \end{cases}$$

Contours of this function (Figure 6) show a central region with a local maximum of 0.75; the function decreasing from each side of the centre along the line $\beta_1 = \beta_2$ to zero values at the corners where $\beta_1 = \beta_2 = 0$ and $\beta_1 = \beta_2 = 1$, and the function increasing from each side of the centre along the line $\beta_1 + \beta_2 = 1$ to unit values at the other two corners. The joint density of X_1, X_2 is simply obtainable but is now a mixture of three, instead of two, components. Thus maximum likelihood estimates of β_1 and β_2 can be obtained.

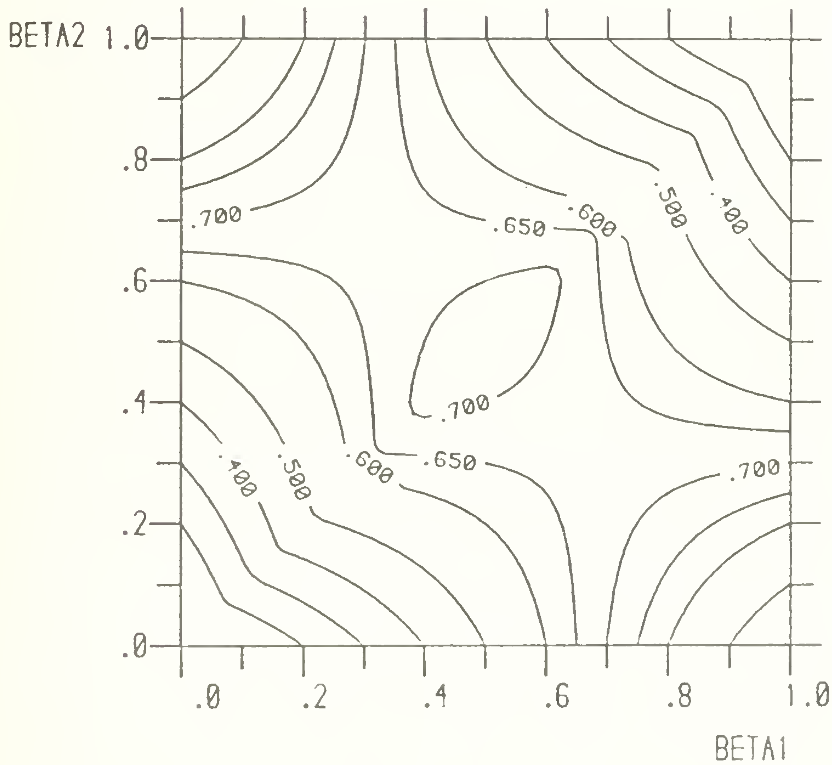

Figure 6. Correlation function of the two-parameter positively dependent EP3(β_1,β_2) model: unlabelled contours may be identified from their joins with the β_1 and β_2 axes.

The edges ($\beta_1 = 0$) or ($\beta_2 = 0$) give a one-parameter bivariate exponential pair which corresponds to adjacent (exponential) variables in an EAR(1) process [Gaver and Lewis, 1980]. Note that using the basic relationship (2.1) outside of the serial constraint posed by a time series gives a broader bivariate exponential pair.

The UP3 Model

The corresponding uniform model, UP3, is given by

$$\text{(UP3)} \qquad Y_1 = 1 - U_1^{\beta_1 I_1} U_2^{I_1}, \quad Y_2 = 1 - U_1^{I_2} U_2^{\beta_2} \qquad (5.4)$$

and has correlation (the Spearman correlation of the EP3 model), given by

$$\operatorname{Corr}(Y_1, Y_2) = 12\left[\frac{1-\beta_2}{(2+\beta_1)(1+\beta_2)} + \frac{\beta_2-\beta_1}{(2+\beta_2)(1+\beta_1)} + \frac{\beta_1}{(1+\beta_1)(1+\beta_2)}\right] - 3 \qquad (0 \leq \beta_1 \leq \beta_2 \leq 1) \ . \qquad (5.5)$$

For $\beta_1 \geq \beta_2$ there is a symmetrical result. The contours of this function represent a deformation of those just described for the EP3 model, symmetry about $\beta_1 = \beta_2$ is preserved, while the area of local maximum is moved towards the origin. Thus, asymmetry in the Spearman dependence is indicated.

The EP4 Model

The negatively dependent version of the EP3 model is the two parameter version of (4.2), and will be called the EP4 model; this has correlation function

$$\mathrm{Corr}(X_1,X_2) = -\,0.6449[\beta_1(1-\beta_2) + \beta_2(1-\beta_1)]$$

$$- \begin{cases} \beta_1\beta_2 & \text{for } \beta_1 + \beta_2 \leq 1 \,, \\ (1-\beta_1)(1-\beta_2) & \text{for } \beta_1 + \beta_2 \geq 1 \,. \end{cases} \tag{5.6}$$

Contours of this function indicate a valley of lowest negative values along the direction $\beta_1 + \beta_2 = 1$, with a central value of -0.5720 and corner values of maximum negativity, -0.6449. For specified negative correlation down to about -0.5 there is a wide choice of (β_1,β_2) pairs, in the two regions either side of the valley along $\beta_1 + \beta_2 = 1$.

The UP4 Model

The final development of the two parameter models is to the negatively dependent two parameter uniform model, the UP4 model. This corresponds to (4.8) with β and I in Y_1 replaced by β_1 and I_1 and β and I in Y_2 replaced by β_2 and $a(I_2)$. Maximum negativity of dependence follows from the 'antithetic' joint distribution for I_1 and I_2 which has $p_{11} = 1 - (\beta_1+\beta_2)$ when $\beta_1 + \beta_2 \leq 1$ and $p_{oo} = \beta_1 + \beta_2 - 1$ when $\beta_1 + \beta_2 \geq 1$. We then have, for the case $\beta_1 + \beta_2 \leq 1$,

$$\mathrm{Corr}(U_1,U_2) = 12\left[\frac{1-(\beta_1+\beta_2)}{(1+\beta_1)(2+\beta_1)(1+\beta_2)(2+\beta_2)} + \frac{\beta_1}{(1+\beta_1)(1+\beta_2)(2+\beta_1)} + \frac{\beta_2}{(1+\beta_1)(1+\beta_2)(2+\beta_2)}\right] - 3\,. \tag{5.7}$$

The corresponding result for $\beta_1 + \beta_2 \geq 1$ is omitted. The contours of this function are quite similar to those of the EP4 model, but with a central

value of -0.867 and associated corner values of minus one; the other two corner values are zero, as for the EP4 model; it is again useful to view these as the Spearman correlations of the EP4 model.

From the point of view of simulation, the two parameter models are hardly more complicated than the one parameter version and offer an extra degree of flexibility. The joint probability density functions and moment generating functions are more complicated than for the one parameter models, but can be derived if needed, for instance, for estimation. It is also possible to derive measures such as $P\{X_1 > X_2\}$, which is not necessarily ½.

6. MORE GENERAL FOUR PARAMETER MODELS: EP+, EP-MODELS

The NEAR(1) exponential time series models of Lawrance and Lewis [1981] suggests a class of four-parameter models; these are likely to be over-complicated for ordinary use, but place the earlier models in a general setting and suggest a further class of two-parameter models. They also give a bivariate exponential pair with full positive correlation, and without the degeneracy which occurs for the case ($\beta_1 = 0$ or $\beta_2 = 0$) in the EP3 model. Being as brief as possible, this development gives the EP+ model as

$$\text{(EP+)} \qquad \begin{aligned} X_1 &= \beta_1 V_1 E_1 + I_1 E_2 \, , \\ X_2 &= \beta_2 V_2 E_2 + I_2 E_1 \, , \end{aligned} \tag{6.1}$$

where, for $i = 1,2$,

$$V_i = \begin{cases} 1 & \text{w.p. } \alpha_i \\ 0 & \text{w.p. } 1-\alpha_i \end{cases} , \qquad I_i = \begin{cases} 1 & \text{w.p. } (1-\beta_i)/[1-(1-\alpha_i)\beta_i] \\ (1-\alpha_i)\beta_i & \text{w.p. } \alpha_i\beta_i/[1-(1-\alpha_i)\beta_i] \end{cases}$$

and the random variables (V_1,V_2), (I_1,I_2) are independent between pairs but usually dependent within pairs. This model has correlation structure of the form

$$\begin{aligned} \text{Corr}(X_1,X_2) &= \beta_1\beta_2 \, \text{Cov}(V_1,V_2) + \text{Cov}(I_1,I_2) \\ &\quad + \alpha_1\beta_1(1-\alpha_2\beta_2) + \alpha_2\beta_2(1-\alpha_1\beta_1) \, . \end{aligned} \tag{6.2}$$

For a model of maximum positive dependency the appropriate joint distributions of (V_1,V_2) and (I_1,I_2) could be obtained using constructions as at (4.4).

A further two parameter model, EP5, is suggested by taking $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $V_1 = V_2$, $I_1 = I_2$; then (6.2) reduces to

$$\mathrm{Corr}(X_1,X_2) = 3\alpha\beta(1-\alpha\beta) \ . \tag{6.3}$$

This has maximum value 0.75 at $\alpha\beta = 0.5$, but does not attain the higher correlations of the EP3 model, and is more complicated in its construction.

Both the EP1 and EP3 models can be obtained as special cases of the EP+ model; the EP1 model is given by taking $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = \beta$ with $I_1 = I_2$, while the EP3 model has $\alpha_1 = \alpha_2 = 1$, (β_1,β_2) unchanged, and with (I_1,I_2) of maximum possible dependency.

Corresponding to (6.1) there is a more general negatively dependent four-parameter model, EP-, of the form

$$\begin{aligned} X_1 &= \beta_1 V_1 E_1 + I_1 E_2 \ , \\ X_2 &= \beta_1 V_1 a(E_2) + I_2 a(E_1) \ , \end{aligned} \tag{6.4}$$

which has, by analogy, the correlation structure

$$\begin{aligned} \mathrm{Corr}(X_1,X_2) &= \beta_1\beta_2 \mathrm{Cov}(V_1,V_2) + \mathrm{Cov}(I_1,I_2) \\ &\quad - 0.6449[\alpha_1\beta_1(1-\alpha_2\beta_2) + \alpha_2\beta_2(1-\alpha_1\beta_1)] \ . \end{aligned} \tag{6.5}$$

It contains the EP2 and EP4 models, and would suggest the further two parameter model, EP6, by taking $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and (V_1,V_2), (I_1,I_2) as two antithetic pairs.

Any further details are omitted, as are the corresponding uniform models of the four-parameter class, UP+, UP- given in Table 1, which summarizes all the models.

7. MULTIVARIATE GENERALIZATIONS

The possibility of multivariate generalizations is apparent by repeated use of the random-coefficient linear functions of (2.1). In fact this is the way the EAR(1) process is constructed with a serial chaining of i.i.d. exponentials $E_1, E_2, \ldots$. Any k variables in an EAR(1) process (or an NEAR(1) process) are a k-variate exponential random variable. Outside of this serial context many possibilities suggest themselves. Thus changing β to β_1, I to I_1 and replacing E_2 by a similar random-coefficient linear function of independent E_2 and E_3 using β_2 and I_2, gives

$$X = \beta_1 E_1 + \beta_2 I_1 E_2 + I_1 I_2 E_3 . \tag{7.1}$$

Here I_1 and I_2 must be independent. A triple of dependent exponential variables could be constructed using E_1, E_2, E_3 in three of the six possible orders.

A simpler possibility would be to use the basic randomly-linear operation on the pairs (E_1, E_2), (E_2, E_3) and (E_3, E_1). Such developments are not considered here. The possibilities are legion but no 'natural' simple method suggests itself above any other.

8. LINE DISCONTINUITIES AND RELIABILITY PROBLEMS

The problem of discriminating amongst the many bivariate exponential models is not simple, although there may be modelling or structural details which recommend certain models in certain contexts. Most of these details are, however, tenuous and difficult to verify from data. An important mathematical scheme is given in Griffiths [1969], who found a canonical expansion for bivariate exponential random variables. Again the models put forward here are simple to generate on a computer and are analytically tractable.

Another property is that the bivariate distributions do not have line discontinuities. However, as an illustration of modelling considerations, we note that in a reliability context thus is not necessarily a virtue. It is known that components in a system can fail from a common cause, which is precisely what gives the line discontinuity in the bivariate distribution. However, it is simple to put this in to the present models and it can be done in at least three ways.

Thus, let (X_1,X_2) denote any unit mean, bivariate exponential pair, let (I_1,I_2) denote an indicator pair, possibly completely or partially dependent, with marginal distributions $P\{I_1 = 1\} = 1 - P\{I_1 = 0\} = P_1$ and $P\{I_2 = 1\} = 1 - P\{I_2 = 0\} = P_2$, and let E be an independent, unit mean exponential random variable. Three new bivariate exponential pairs are given

(i) $$Z_1 = I_1X_1 + (1-I_1)E\ , \qquad Z_2 = I_2X_2 + (1-I_2)E\ ; \tag{8.1}$$

(ii) $$Z_1 = \beta E + I_1X_1\ , \qquad Z_2 = \beta E + I_2X_2\ ; \tag{8.2}$$

and

$$(iii) \quad Z_1 = \min(X_1, E) , \qquad Z_2 = \min(X_2, E) \qquad (8.3)$$

In all three cases there is a non-zero probability that Z_1 and Z_2 are proportional to E .

The first pair uses the idea that mixtures of identically distributed random variables have that same distribution, the second pair uses the basi relationship (2.1) and the third uses the fact that the minimum of independent exponential random variables is an exponential random variable. In fact, if X_1 and X_2 are independent, then (8.3) is the Marshall-Olkin model.

ACKNOWLEDGMENTS

The work of Professor P. A. W. Lewis was supported by the Office of Naval Research under Grant NR-42-469.

TABLE I

Summary of Models Considered (See text for relevant details)

EP1	$X_1 = \beta E_1 + IE_2$ $X_2 = \beta E_2 + IE_1$	UP1	$Y_1 = 1-U_1^{\beta}U_2^{I}$ $Y_2 = 1-U_1^{I}U_2^{\beta}$
EP2	$X_1 = \beta E_1 + I_1E_2$ $X_2 = \beta a(E_2) + I_2a(E_1)$	UP2	$Y_1 = 1-U_1^{\beta}U_2^{I}$ $Y_2 = 1-(1-U_1)^{a(I)}(1-U_2)^{\beta}$
EP3	$X_1 = \beta_1E_1 + I_1E_2$ $X_2 = \beta_2E_2 + I_2E_1$	UP3	$Y_1 = 1-U_1^{\beta_1}U_2^{I}$ $Y_2 = 1-U_1^{I_2}U_2^{\beta_2}$
EP4	$X_1 = \beta_1E_1 + I_1E_2$ $X_2 = \beta_2a(E_2) + I_2a(E_1)$	UP4	$Y_1 = 1-U_1^{\beta_1}U_2^{I}$ $Y_2 = 1-(1-U_1)^{I_2}(1-U_2)^{\beta_2}$
EP+	$X_1 = \beta_1V_1E_1 + I_1E_2$ $X_2 = \beta_1V_2E_2 + I_2E_1$	UP+	$Y_1 = 1-U_1^{\beta_1V_1}U_2^{I_1}$ $Y_2 = 1-U_1^{I_2}U_2^{\beta_2V_2}$
EP-	$X_1 = \beta_1V_1E_1 + I_1E_2$ $X_2 = \beta_1V_2a(E_2) + I_2a(E_1)$	UP-	$Y_1 = 1-U_1^{\beta_1V_1}(1-U_2)^{I_1}$ $Y_2 = 1-(1-U_1)^{I_2}(1-U_2)^{\beta_1V_2}$

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